CONCERNING THE ORDER OF APPROXIMATION OF PERIODIC CONTINUOUS FUNCTIONS BY TRIGONOMETRIC INTERPOLATION POLYNOMIALS

BY

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ABSTRACT

Let $x_{kn} = 2\pi k/n$, $k = 0, 1 \ldots n-1$ (n odd positive integer). Let $R_n(x)$ be the unique trigonometric polynomial of order $2n$ satisfying the interpolatory conditions: $R_n(x_{kn}) = f(x_{kn})$, $R_n^{(j)}(x_{kn}) = 0$, $j = 1, 2, 4$, $k = 0, 1 \ldots, n-1$. We set $w_2(t,f)$ as the second modulus of continuity of $f(x)$. Then we prove that $|R_n(x) - f(x)| = o(nw_2(1/nf))$. We also examine the question of lower estimate of $\|R_n - f\|$. This generalizes an earlier work of the author.

Let

$$x_{kn} = \frac{2\pi k}{n}, \; (k = 0, 1, \cdots, n-1;\; n \text{ is an odd positive integer}). \tag{1.1}$$

Let $R_n(x)$ be the unique trigonometric polynomial of order $2n$ determined by the interpolatory conditions

$$R_n(x_{kn}) = f(x_{kn}),\; R_n^{(j)}(x_{kn}) = 0, j = 1, 2, 4, k = 0, 1, \cdots, n-1. \tag{1.2}$$

We assume that $R_n(x)$ has the following form

$$d_0 + \sum_{j=1}^{2n-1} (d_j \cos jx + e_j \sin jx) + d_{2n} \cos 2nx. \tag{1.3}$$

In our earlier work [3], we considered the problem of existence, uniqueness, explicit representation and problem of uniform convergence of $R_n(x)$ to $f(x)$ on the real line. The main theorem of [3] is as follows.

THEOREM 1. *a) If $f(x)$ is a 2π periodic continuous function satisfying the Zygmund condition λ*

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$$(1.4)\qquad |f(x+h)-2f(x)+f(x-h)| = o(h),$$

then the sequence $\{R_n(x)\}$ *defined by* (1.2) *and* (1.3) *converges uniformly to* $f(x)$ *over the real line. b) Further, Zygmund class cannot be replaced by Lip* α, $0<\alpha<1$.

The objects of this paper is to examine the question of lower and upper estimates of $\|f-R_n(f)\|$ in uniform norm for a certain class of continuous periodic functions.

THEOREM 2. *Let* $w_2\,(t,f)$ *be the second modulus of continuity of* $f(x)$. *Then we have*

$$(1.5)\qquad |\,R_n(x)-f(x)| = o\left(n\,w_2\left(\frac{1}{n},f\right)\right).$$

COROLLARY. *Let* $f(x)$ *satisfy* (1.4). *From* (1.5) *we have* $|\,R_n(x)-f(x)|=o(1)$. *Thus Theorem* 1–*a*) *is a special case of Theorem* 2.

Let us denote by $\tilde{c}(\phi)$ the class of all 2π periodic continuous functions for which

$$(1.6)\qquad w_2(t,f) = 0(\phi(t)).$$

Let $\phi(t)$ have the following properties

$$(1.7)\qquad \begin{cases} \text{i)} & \phi(t)>0 \text{ for } t>0,\ \phi(0)=0,\ \phi(T)\geqq\phi(t),\ T\geqq t,\\ \text{ii)} & \phi(t) \text{ is continuous for } t>0,\\ \text{iii)} & t^2/\phi(t) \text{ is monotonic increasing for } t\geqq 0\\ \text{iv)} & \lim_{t\to 0+} t^2/\phi(t)=0. \end{cases}$$

THEOREM 3. *There exists a* 2π *periodic continuous function* f *belonging to* $\tilde{c}(\phi)$ *for which*

$$(1.8)\qquad |\,R_n(\pi)-f(\pi)| > c\,n\,\phi\left(\frac{1}{n}\right) \text{ for } n=n_1,\,n_2,\cdots$$

where $0<n_1<n_2<\cdots$ *and* n *is always an odd positive integer.*

COROLLARY. *There exists a continuous* 2π *periodic function from the class of functions which satisfy the Zygmund condition* $w_2(f,h)=O(h)$ *such that* $|\,R_n(\pi)-f(\pi)|>c$, *i.e., for such a function,* $R_n(x)$ *cannot converge uniformly to* $f(x)$ *on the real line.*

Thus Theorem 3 is much stronger than Theorem 1(b).

Proof of Theorem 2 depends on the following

THEOREM 4 (*S. B. Steckin*). *Let k be a positive integer. Then there exists a positive constant c_k such that for every $f \in c_{2\pi}$ we can find a trigonometric polynomial of order n at most such that*

$$\|f - t_n\| \leqq c_k w_k\left(\frac{1}{n}, f\right) \tag{2.1}$$

and

$$\| t_n^{(k)} \| \leqq B_k n^k w_k\left(\frac{1}{n}, f\right). \tag{2.2}$$

Here $w_k(\delta, f)$ is the modulus of smoothness of order k of $f(x)$.

From (2.2) we have for $k = 2$

$$\| t_n^{(2)} \| \leqq B_2 n^2 w_2\left(\frac{1}{n}, f\right). \tag{2.3}$$

On using the Bernstein inequality twice, we have

$$\| t_n^{(iv)} \| \leqq B_2 n^4 w_2\left(\frac{1}{n}, f\right). \tag{2.4}$$

Using a similar approach as in [2] one can prove

$$\| t_n^{(1)} \| \leqq B_2 n^2 w_2\left(\frac{1}{n}, f\right). \tag{2.5}$$

In fact one can prove more than (2.5) but, for our purposes, this is sufficient. Now we prove Theorem 2.

PROOF OF THEOREM 2. From [3] it follows that for $n = 1, 3, 5, \cdots$ we have

$$\begin{aligned} f(x) - R_n(x) = f(x) - t_n(x) &+ \sum_{k=0}^{n-1} (t_n(x_{kn}) - f(x_{kn}))A(x - x_{kn}) \\ &+ \sum_{k=0}^{n-1} t_n'(x_{kn})B(x - x_{kn}) + \sum_{k=0}^{n-1} t_n''(x_{kn})C(x - x_{kn}) \\ &+ \sum_{k=0}^{n-1} t_n^{(iv)}(x_{kn})D(x - x_{kn}). \end{aligned} \tag{2.6}$$

Here $A(x - x_{kn})$, $B(x - x_{kn})$, $C(x - x_{kn})$ and $D(x - x_{kn})$ are fundamental functions of $(0, 1, 2, 4)$ interpolation. Their explicit forms and estimates are given in our earlier work [3]. Here we need only use the lower and upper estimates.

$$\begin{aligned}
&\sum_{k=0}^{n-1} | A(x - x_{kn}| \leqq 11n, \quad \sum_{k=0}^{n-1} | A(\pi - x_{kn})| > c_3 n, \\
&\sum_{k=0}^{n-1} | B(x - x_{kn})| \leqq \frac{3}{n}, \quad \sum_{k=0}^{n-1} B\left(\frac{\pi}{2} - x_{kn}\right) > \frac{c_4}{n}, \\
&\sum_{k=0}^{n-1} | C(x - x_{kn})| \leqq \frac{13}{n}, \quad \sum_{k=0}^{n-1} | C(\pi - x_{kn}) > \frac{c_3}{n}, \\
&\sum_{k=0}^{n-1} | D(x - x_{kn})| \leqq \frac{2\pi}{3n^3}, \quad \sum_{k=0}^{n-1} | D(\pi - x_{kn})| > \frac{1}{n^3}.
\end{aligned} \tag{2.7}$$

On using (2.1)–(2.5) and (2.7) we have

$$\begin{aligned}
|f(x) - R_n(x)| &\leqq c_2 w_2\left(\frac{1}{n}, f\right) + c_2 w_2\left(\frac{1}{n}, f\right) 11n \\
&+ B_2 n^2 w_2\left(\frac{1}{n}, f\right) \frac{3}{n} + B_2 n^2 w_2\left(\frac{1}{n}, f\right) \frac{13}{n} \\
&+ B_2 n^4 w_2\left(\frac{1}{n}, f\right) \frac{2\pi}{3n^3} \\
&\leqq cn w_2\left(\frac{1}{n}, f\right),
\end{aligned}$$

where $c = \max\{c_2, 13, B_2\}$. This proves Theorem 2.

Proof of Theorem 3 is a direct application of a recent result of O. Kis and P. Vertesi [1].

Let x_{kn}, $n = 1, 2, \cdots$ be an infinite point system such that $0 \leqq x_{kn} < 2\pi$. We define

$$L_n(f, x) = \sum_{k=0}^{n-1} f(x_{kn}) P_{kn}(x),$$

$$\lambda_n(x) = \sum_{k=0}^{n-1} | P_{kn}(x)|$$

where $P_{kn}(x)$ are 2π periodic continuous functions.

THEOREM 5 (*O. Kis and P. Vertesi*). *If* $-\infty < x_0 < \infty$ *and* $\lim_{n\to\infty} \lambda_n(x_0) \neq 1$, *then there exist* $f(x), f \in \tilde{c}(\phi)$ *and integers* $0 < n_1 < n_2 < \cdots$ *such that* $|f(x_0) - L_{n_k}(f, x_0)| > \lambda_{n_k}(x_0)\phi(d_{n_k})$ *for* $k = 1, 2, \cdots$. *Here* $d_n = \min(x_{k+1,n} - x_{kn})$.

NOTE. The Theorem is valid even in the case where $\lim_{n\to\infty} \lambda_n(x_0)$ does not exist.

PROOF OF THEOREM 3. We choose here $p_{nk}(x) = A(x - x_{kn})$, $x_0 = \pi$, $d_n = 2\pi/n$, $n = 1, 3, 5, 7 \ldots..$ and observe that $\lambda_n(\pi) = \sum_{k=0}^{n-1} | A(\pi - x_{kn})| > c_3 n$ from (2.7). The conclusion of our theorem follows immediately.

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